Transformations for an Accelerated Observer in Special Relativity

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Abstract

We want to understand how the world behaves when you accelerate, in the context of Special Relativity. We want to do something like the Lorentz Transformation, except for acceleration and not velocity. In this article, we develop equations for the velocity, acceleration and force of an object as they are measured in an accelerated frame. Using these equations, it is then possible to completely analyze cases such as the “twin paradox”. The ability to analyze such cases is indispensable for Special Relativity research.

Contents

1. Introduction and first example

We want to understand how the world behaves when you accelerate. We want to do something like the Lorentz Transformation, except for acceleration and not velocity. The Lorentz Transformation, in its simplest one-dimensional form, is a transformation between an observer at rest, with zero velocity and zero acceleration, and an observer with a velocity v along the x-axis. Both observers are at $(position = 0, time = 0)$ according to both their reference frames. Our transformation will be between an observer at rest, also with zero velocity and zero acceleration, and an observer with acceleration $\tilde{a}$ on the x-axis. The first observer will be an inertial observer, the second one will be a non-inertial observer. Our accelerated observer will have zero velocity. And as in the Lorentz Transformation, in ours too both observes will be at $(pos = 0, time = 0)$ according to both their reference frames. We will dub the inertial observer as O, and the accelerating one as O'.

Assumption 1: O and O' will agree about the coordinates of all events taking place at $time = 0$. That means that they will agree on the locations of all objects and the time reading on all clocks in the system, at $time = 0$.

Note that this is an assumption that does not hold in the case of the Lorentz Transformation.

Before we start with the simplest example, I want to point out a difference between our transformation and the Lorentz Transformation. The Lorentz Transformation is concerned with space-time events, not with objects. Our transformation will deal with objects: How their velocities, accelerations, masses etc. change when an observer is accelerating.

Let's start with the simplest example. There is an object, let's call it Obj. According to O, the object is stationary at $pos = d$. It has zero velocity and zero acceleration at all times. We want to know what O' would think are the velocity and acceleration of Obj. (Note: we already know that the position of Obj in the O' frame is d, according to Assumption 1.)

How do we find out what are the velocity and acceleration of Obj in the O' reference frame? This is the answer: We can know what is the position of Obj in the O' frame at any point in time, using the Lorentz Transformation and some other manipulations. We will obtain the velocity and acceleration of Obj using the fundamental definitions: Velocity as a derivative of position over observer time, and acceleration as derivative of velocity over observer time.

Now our mission: To obtain an expression of the position of Obj in the O' frame, as a function of time in that frame.

To do this, we are going to look at things not from the O' frame, but from the O frame, at least at first. Assume some time t has passed in the O frame. What has happened with O'? He is now at $\frac{1}{2}\tilde{a}t^2$, and his velocity is $\tilde{a}t$. How much time passed in his frame? That will be a harder calculation. Because O' has been moving at some non-zero speed, his clock has been running slower than the clock of O. How much slower? We will calculate.

This is the momentary ratio of time dilation of O':

$$\sqrt{1-\frac{(\tilde{a}t)^2}{c^2}} \tag{1}$$

So if we want to know what is the reading on the clock of O' at $time = t$, we will use this function:

$$exp(t) = \int_0^t \sqrt{1-\frac{(\tilde{a}t)^2}{c^2}}\; dt \tag{2}$$

Which simplifies to:

$$exp(t) = \frac{\tilde{a}t\sqrt{1-\frac{\tilde{a}t^2}{c^2}} + c\arcsin\left(\frac{\tilde{a}t}{c}\right)}{2\tilde{a}} \tag{3}$$

Note: You may have noticed that the above formulas "break" when t grows too big. That is correct. These formulas are only intended for small values of t. We are only interested in small time values because our plan is to differentiate various expressions over time.

Ok, now we know how much time passed in the O' frame. Now we need to find out the position of Obj in the O' frame. For this, we are going to create two more reference frames, that we will call T and T', since they are just temporary aids for our calculations. Remember that now we are analyzing the system at $time = t$, as measured in the O frame. The observer T is at rest according to O, and the observer T' is at rest according to O'. The observers T and T' are both in the same position: Exactly where O' is. To remind you, that position is $\frac{1}{2}\tilde{a}t^2$. Both T and T' consider the time to be zero, while O considers the time to be t. Do you see where this is going? This is our plan for getting the position of Obj in the O' system: We have the position of Obj in the O frame. A Galilean transformation will give us the position in the T frame. A Lorentz transformation will give us the position in the T' frame. A Galilean transformation will give us the position in the O' frame. The position in the O frame is d. Via Galilean, the position in T is $d - \frac{1}{2}\tilde{a}t^2$. Now to do a Lorentz transformation to T'. The velocity of T' is like that of O': $\tilde{a}t$. Therefore the position in T' is $\left(d - \frac{1}{2}\tilde{a}t^2\right)\sqrt{1 - \left(\frac{\tilde{a}t}{c}\right)^2}$. When we do a Galilean transformation to O', the position stays the same, only the time coordinate changes. Now we have it, the position and time coordinates of Obj in the O' system: $(pos = \left(d - \frac{1}{2}\tilde{a}t^2\right)\sqrt{1 - \left(\frac{\tilde{a}t}{c}\right)^2},\ time = exp(t))$.

Now remember what is the purpose of this: To get the velocity and acceleration of Obj in $time = 0$. This is the expression for the position when $time = exp(t)$:

$$pos(t) = \left(d - \frac{1}{2}\tilde{a}t^2\right)\sqrt{1 - \left(\frac{\tilde{a}t}{c}\right)^2} \tag{4}$$

We differentiate this function with respect to $exp(t)$ to get a velocity function, which I will not duplicate here since it is more cumbersome than illuminating. The important thing is its value at $t = 0$:

$$vel(0) = 0 \tag{5}$$

This means that O' agrees with O that the object has no velocity.

We obtain an acceleration function by differentiating again. This is its value at $t = 0$:

$$acc(0) = -\tilde{a}\left(1 + \frac{\tilde{a}d}{c^2}\right) \tag{6}$$

This is the first interesting result. In a Newtonian world, the acceleration would be $-\tilde{a}$, but here there is a factor $(1 + \frac{\tilde{a}d}{c^2})$. This is an important factor that will appear in almost all of the expressions we will find.

Now I want to know what the mass m' of Obj is in the O' frame, given that it's rest mass is m_0. (In this example, the mass of Obj in the O frame, dubbed m, is identical to the rest mass m_0. In following examples this will not be the case.) The mass factor m'/m_0 is identical to the time dilation, and this is how we will measure it. The experiment begins at $time = 0$. We let t time pass in the O system. The observer O' experienced $exp(t)$ time. Using the Lorentz transformation, we will find out how much time passed in the clock on Obj according to O'. We will divide $exp(t)$ by that time. We will get an expression that depends on t. We will take its limit as $t \to 0$. The result will be m'/m_0.

We will use the "virtual" observers that we created before: T and T'. But now the Lorentz transformation will be a little more complicated. Obj is an object. Therefore, Obj has a world-line, which is a set of space-time events. Since Obj is stationary in our case, this would be a general space-time event in the world-line of Obj, as described by O: $(pos = d,\ time = \tau)$. (We are using τ as our free variable since t is taken.) Now we will transform this event into T with a Galilean transformation: $(pos = d - \frac{1}{2}\tilde{a}t^2,\ time = \tau - t)$. Now Lorentz into T': $(pos = \gamma_{\tilde{a}t}(d - \frac{1}{2}\tilde{a}t^2 - \tilde{a}t(\tau - t)),\ time = \gamma_{\tilde{a}t}(\tau - t - \frac{\tilde{a}t(d - \frac{1}{2}\tilde{a}t^2)}{c^2}))$. Now Galilean into O': $(pos = \gamma_{\tilde{a}t}(d - \frac{1}{2}\tilde{a}t^2 - \tilde{a}t(\tau - t)),\ time = \gamma_{\tilde{a}t}(\tau - t - \frac{\tilde{a}t(d - \frac{1}{2}\tilde{a}t^2)}{c^2}) + exp(t))$. Now we want the event for which $time = exp(t)$ in the O' frame. Therefore, we need to solve the following equation for τ:

$$\gamma_{\tilde{a}t}\left(\tau - t - \frac{\tilde{a}t\left(d - \frac{1}{2}\tilde{a}t^2\right)}{c^2}\right) + exp(t) = exp(t) \tag{7}$$

This is a linear equation, and its solution is:

$$\tau = t + \frac{\tilde{a}dt}{c^2} - \frac{\tilde{a}^2t^3}{2c^2} \tag{8}$$

So this is our τ. Since Obj is non-moving, τ is exactly the time that passed on Obj's clock. Now we will divide the time experienced by O' by the time experienced by Obj to get the mass of Obj:

$$m' = m_0 \lim_{t\to 0} \frac{exp(t)}{\tau} = m_0\left(1 + \frac{\tilde{a}d}{c^2}\right)^{-1} \tag{9}$$

Again the same expression appears: The mass factor is $(1 + \frac{\tilde{a}d}{c^2})^{-1}$. We have obtained the mass.

2. An objectionable result

Think about that expression for the mass and how it behaves in different circumstances. When you accelerate towards an object, time on that object will run faster. The farther away the object is, and the more you accelerate, the faster its "fast forward" will be. When you are accelerating away from an object, there are two cases. If the object is sufficiently close to you, time on it will run slower. The faster your acceleration is and the farther away the object is, the slower his "slow motion" will become. But, if the object his sufficiently far behind you, the factor $(1 + \frac{\tilde{a}d}{c^2})^{-1}$ will become negative! What this means is that the mass of the object will be negative and it will be going back in time.

This is a provocative and objectionable result. However, I do not see how it could be denied. True, it sounds crazy; It is strongly opposed to intuition. But when dealing with

Special Relativity, we should be skeptical about what our intuition tells us. For example, according to Special Relativity, if I get up from my chair and run, the tree in the garden will become heavier. So we shouldn't dismiss a result just because it sounds crazy.

What more objections are there to this result? It seems to contradict the second law of Thermodynamics. For example, if the aforementioned object is a room in which a person throws an egg on the floor, then according to us the egg has reassembled itself and jumped back into the person's hand. This seems to imply that the entropy in the room had declined, which would be forbidden by the second law. However, the second law is a "high-level" law. It is based on laws of mechanics which are on a low abstraction level. The provocative result that we have obtained came directly from the level of mechanics. Therefore, if there is a contradiction, it should appear on the level of mechanics as well.

I do not believe that the second law is wrong; I think that in order to apply it when the observer is accelerating, it is necessary to generalize it. I have not given this matter much thought, but I think that we can say that the second law applies only when the mass is positive, and when the mass is negative a negative version of the second law is needed.

What more objections are there to our result? On first glance, it seems that it might lead to paradox. If some things in our universe are going back in time, can we not obtain information from their future and deliver it to their past? The answer is no. The equations for this "reverse motion" behavior forbid us from doing that. Remember that an object will be going back in time only if it's far enough behind you? The point is that it's always just too far for you to "exploit" the reverse motion to create a paradox. After I will give you the generalized equations, you are welcome to try and construct a paradox using these laws. It will become clear that even though some objects are going backwards in time, there will be no breaking of causality laws and no opportunity for "time travel".

3. An object moving at a constant velocity

Let's continue. We have solved the problem of a stationary object. Now we will solve the problem of a constant-velocity moving object. We will call the velocity of the object v.

In this point I will have to introduce another complication. Up to this point we assumed that the acceleration of O' is constant. Indeed, in the results we have obtained so far it does not matter whether the acceleration is constant or wildly fluctuating. However, from now on it will sometimes matter. Therefore some of our results will depend not only on the acceleration of O', but also on the derivative of the acceleration. The derivative of the acceleration, which is also the third derivative of the position, is called "jerk". We will refer to the jerk of O' as $\tilde{\jmath}$. Sometimes we will take the jerk into consideration, and sometimes we won't, depending on whether it will matter in the final result.

A general space-time event on Obj's worldline will not be $(pos = d,\ time = \tau)$ as it was before; It will now be $(pos = d + v\tau,\ time = \tau)$. Galilean into T: $(pos = d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3,\ time = \tau - t)$ (notice how we took the jerk into account?). Lorentz into T':

$$\begin{aligned} (pos &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)), \\ time &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\tau - t) - \frac{\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2}{c^2}(d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3))) \end{aligned} \tag{10}$$

Galilean into O':

$$\begin{aligned} (pos &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)) \\ time &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\tau - t) - \frac{\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2}{c^2}(d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3)) + exp(t)) \end{aligned} \tag{11}$$

This is the equation to be solved:

$$\gamma_{\tilde{a}t}\left((\tau - t) - \frac{\tilde{a}t}{c^2}\left(d + v\tau - \frac{1}{2}\tilde{a}t^2\right)\right) + exp(t) = exp(t) \tag{12}$$

This is the solution:

$$\tau = \frac{t \cdot \left(-12c^2 + \left(2\tilde{a} + \tilde{\jmath}t\right)\left(-6d + t^2\left(3\tilde{a} + \tilde{\jmath}t\right)\right)\right)}{6 \cdot \left(-2c^2 + t \cdot v \cdot \left(2\tilde{a} + \tilde{\jmath}t\right)\right)} \tag{13}$$

This is the position of the space-time event in the frame of O', dependent on τ:

$$\gamma_{(\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)}((d + v\tau - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)) \tag{14}$$

We enter the solution we found for τ and simplify to get the position function for Obj, dependent on t. Don't try to make sense of it, it is not so interesting by itself:

$$pos(t) = \frac{c^2 \gamma^{-1}_{(\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)}(-6d + t \cdot (3\tilde{a}t + \tilde{\jmath}t^2 - 6v))}{3 \cdot (-2c^2 + t \cdot v \cdot (2\tilde{a} + \tilde{\jmath}t))} \tag{15}$$

We differentiate once and twice with respect to $exp(t)$ in order to get the velocity and acceleration functions. At $time = 0$, they are:

$$vel(0) \quad \rightarrow \quad v\left(1 + \frac{\tilde{a}d}{c^2}\right) \tag{16}$$

$$acc(0) \quad \rightarrow \quad \left(1 + \frac{\tilde{a}d}{c^2}\right)\left(-\tilde{a} + 2\tilde{a} \cdot \frac{v^2}{c^2}\right) + \frac{\tilde{\jmath}d}{c^2}v \tag{17}$$

So we have the velocity and the acceleration. The velocity expression is very satisfactory, employing our favorite factor again. The acceleration is messier, though. Notice that the jerk appears in it.

Now we will obtain the mass.

$$m' = m_0 \lim_{t \to 0} \frac{exp(t)}{\tau\gamma_v^{-1}} = m_0\gamma_v\left(1 + \frac{\tilde{a}d}{c^2}\right)^{-1} = m\left(1 + \frac{\tilde{a}d}{c^2}\right)^{-1} \tag{18}$$

This is very nice! The mass factor due to velocity, gamma, and the mass factor due to acceleration, $(1 + \frac{\tilde{a}d}{c^2})^{-1}$, are just multiplied together! It's a good thing that they are combined in such a simple way.

4. An object moving at a constant acceleration

Now it's time to take on an object going at a constant acceleration. We will dub the object's acceleration a, and its starting velocity v. Now we will also need a function to tell us the time that Obj experienced. This is it:

$$oexp(t) = \int_0^t \sqrt{1 - \left(\frac{at+v}{c^2}\right)^2} \tag{19}$$

This is it with the integral solved:

$$oexp(t) = \frac{c \arcsin\left(\frac{at+v}{c}\right) + (at+v)\gamma_{at+v}^{-1} - c\arcsin\left(\frac{v}{c}\right) - v\gamma_v^{-1}}{2a} \tag{20}$$

Let's do our drill. General space-time event in Obj's worldline in the O system: $(pos = \frac{1}{2}a\tau^2 + v\tau + d,\ time = \tau)$. Galilean to T: $(pos = \frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3,\ time = \tau - t)$. Lorentz to T':

$$\begin{aligned}
(pos &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)),\\
time &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\tau - t) - \frac{\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2}{c^2}(\frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3)))
\end{aligned} \tag{21}$$

Galilean to O':

$$\begin{aligned}
(pos &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)),\\
time &= \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\tau - t) - \frac{\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2}{c^2}(\frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3)) + exp(t))
\end{aligned} \tag{22}$$

This is the equation we need to solve for τ:

$$\gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}((\tau - t) - \frac{\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2}{c^2}(\frac{1}{2}a\tau^2 + v\tau + d - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3)) + exp(t) = exp(t) \tag{23}$$

Now we have a bigger challenge than in the previous examples. Now we have a quadratic equation with two solutions to choose from. We will choose the following solution, for reasons too obscure to explain here. This is it:

$$\tau = \frac{-6c^2 + 3vt(2\tilde{a} + \tilde{\jmath}t) + \sqrt{3}\sqrt{12c^4 - 12c^2t(2\tilde{a} + \tilde{\jmath}t)(at + v) + t^2(2\tilde{a} + \tilde{\jmath}t)^2(a(-6d + t^2(3\tilde{a} + \tilde{\jmath}t)) + 3v^2)}}{-3at(2\tilde{a} + \tilde{\jmath}t)} \tag{24}$$

To obtain the position function $pos(t)$, we will take the position coordinate from equation 22, and substitute the preceding solution for τ. I will not list this function here because it is very cumbersome.

We differentiate $pos(t)$ once and twice with respect to $exp(t)$ to get velocity and acceleration:

$$vel(0) \to v\left(1 + \frac{\tilde{a}d}{c^2}\right)$$

We can see that the velocity did not change, thus it does not depend upon the acceleration of the object.

Now the acceleration:

$$acc(0) \to \left(1 + \frac{\tilde{a}d}{c^2}\right)\left[\left(1 + \frac{\tilde{a}d}{c^2}\right)a + 2\frac{v^2}{c^2}\tilde{a} - \tilde{a}\right] + \frac{\tilde{\jmath}d}{c^2}v \tag{25}$$

The acceleration is an interesting new equation.

Let's get the mass. For that, we will use our newly-defined function $oexp(t)$. This is the mass:

$$m' = m_0 \lim_{t\to 0}\frac{exp(t)}{oexp(t)} = m_0\gamma_v\left(1 + \frac{\tilde{a}d}{c^2}\right)^{-1} = m\left(1 + \frac{\tilde{a}d}{c^2}\right)^{-1} \tag{26}$$

We can see that the mass doesn't depend on the acceleration either.

Now we will do something that we haven't done in the previous problems. We will find the transformation function for the force acting upon the object. Since the force is just the derivative of the momentum, this should not be a problem. Now we will have to figure out the functions for the mass and momentum of the object.

Now we need a function that will express the mass of the object in $time = t$. We will obtain this by using the equation for mass that we just discovered. However, we will have to apply a Lorentz Transformation on it for it to tell us the mass at arbitrary time t. The mathematics is quite complicated, but this is the final result:

$$mass(t) = m_0 \frac{1}{\left(1 + \frac{\tilde{a}\gamma_{\tilde{a}t}^3 pos(t)}{c^2}\right)\sqrt{1 - \left(\frac{vel(t)}{\left(1+\frac{\tilde{a}\gamma_{\tilde{a}t}^3 pos(t)}{c^2}\right)}\frac{1}{c}\right)^2}} \tag{27}$$

This is the momentum:

$$mom(t) = mass(t)\ vel(t) \tag{28}$$

And at $time = 0$:

$$mom(0) \to m_0 \gamma_v v = mv \tag{29}$$

We can see that the expression for the momentum does not depend even on the acceleration of the observer! Hence when an observer accelerates, his opinion of objects' momenta does not change.

Now this would be the force on Obj:

$$force(t) = \lim_{h \to 0} \frac{mom(t+h) - mom(t)}{exp(t+h) - exp(t)} \tag{30}$$

Which evaluates to:

$$force(0) \to m_0 \gamma_v^3 \left(\left(1 + \frac{\tilde{a}d}{c^2}\right) a - \gamma_v^{-2} \tilde{a} \right) \tag{31}$$

Notice something interesting: The jerk $\tilde{j}$ does not appear in the force expression. This is surprising, because it appears in the expression for acceleration.

5. Generalizing to three dimensions

So far we've been dealing with a one-dimensional world. Now we will generalize our findings to a three-dimensional world.

General space-time event in the O frame:

$$\left(x = \frac{1}{2}a_x\tau^2 + v_x\tau + d_x,\ y = \frac{1}{2}a_y\tau^2 + v_y\tau + d_y,\ z = \frac{1}{2}a_z\tau^2 + v_z\tau + d_z,\ time = \tau\right) \tag{32}$$

Galilean to T:

$$\left(x = \frac{1}{2}a_x\tau^2 + v_x\tau + d_x - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3,\ y = \frac{1}{2}a_y\tau^2 + v_y\tau + d_y,\ z = \frac{1}{2}a_z\tau^2 + v_z\tau + d_z,\ time = (\tau - t)\right) \tag{33}$$

Lorentz to T':

$$\Bigg(x = \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}\left(\left(\frac{1}{2}a_x\tau^2 + v_x\tau + d_x - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3\right) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)\right),\ y = \frac{1}{2}a_y\tau^2 + v_y\tau + d_y,$$
$$z = \frac{1}{2}a_z\tau^2 + v_z\tau + d_z,\ time = \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}\left((\tau - t) - \frac{(\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)}{c^2}\left(\frac{1}{2}a_x\tau^2 + v_x\tau + d_x - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3\right)\right)\Bigg) \tag{34}$$

Galilean to O':

$$\Bigg(x = \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}\left(\left(\frac{1}{2}a_x\tau^2 + v_x\tau + d_x - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3\right) - (\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)(\tau - t)\right),\ y = \frac{1}{2}a_y\tau^2 + v_y\tau + d_y,$$
$$z = \frac{1}{2}a_z\tau^2 + v_z\tau + d_z,$$
$$time = \gamma_{(\tilde{a}t+\frac{1}{2}\tilde{\jmath}t^2)}\left((\tau - t) - \frac{(\tilde{a}t + \frac{1}{2}\tilde{\jmath}t^2)}{c^2}\left(\frac{1}{2}a_x\tau^2 + v_x\tau + d_x - \frac{1}{2}\tilde{a}t^2 - \frac{1}{6}\tilde{\jmath}t^3\right)\right) + exp(t)\Bigg) \tag{35}$$

The equation to solve and the resulting τ are the same as those in the previous example.

$$\tau = \frac{-6c^2 + 3vt(2\tilde{a} + \tilde{\jmath t}) + \sqrt{3}\sqrt{12c^4 - 12c^2t(2\tilde{a} + \tilde{\jmath t})(at + v) + t^2(2\tilde{a} + \tilde{\jmath t})^2(a(-6d + t^2(3\tilde{a} + \tilde{\jmath t})) + 3v^2)}}{-3at(2\tilde{a} + \tilde{\jmath t})} \tag{36}$$

I will skip writing down the position functions. This is the velocity at $time = 0$:

$$vel_x(0) \rightarrow v_x\left(1+\frac{\tilde{a}d_x}{c^2}\right) \tag{37}$$

$$vel_y(0) \rightarrow v_y\left(1+\frac{\tilde{a}d_x}{c^2}\right) \tag{38}$$

$$vel_z(0) \rightarrow v_z\left(1+\frac{\tilde{a}d_x}{c^2}\right) \tag{39}$$

$$\vec{vel}(0) \rightarrow \vec{v}\left(1+\frac{\tilde{a}d_x}{c^2}\right) \tag{40}$$

This is the acceleration:

$$acc_x(0) \rightarrow \left(1+\frac{\tilde{a}d_x}{c^2}\right)\left[a_x\left(1+\frac{\tilde{a}d_x}{c^2}\right)+\tilde{a}\left(2\frac{v_x^2}{c^2}\right)-\tilde{a}\right]+\frac{\tilde{j}d_x}{c^2}v_x \tag{41}$$

$$acc_y(0) \rightarrow \left(1+\frac{\tilde{a}d_x}{c^2}\right)\left[a_y\left(1+\frac{\tilde{a}d_x}{c^2}\right)+\tilde{a}\left(2\frac{v_x v_y}{c^2}\right)\right]+\frac{\tilde{j}d_x}{c^2}v_y \tag{42}$$

$$acc_z(0) \rightarrow \left(1+\frac{\tilde{a}d_x}{c^2}\right)\left[a_z\left(1+\frac{\tilde{a}d_x}{c^2}\right)+\tilde{a}\left(2\frac{v_x v_z}{c^2}\right)\right]+\frac{\tilde{j}d_x}{c^2}v_z \tag{43}$$

$$a\vec{c}c(0) \rightarrow \left(1+\frac{\tilde{a}d_x}{c^2}\right)\left[\vec{a}\left(1+\frac{\tilde{a}d_x}{c^2}\right)+\vec{v}\left(2\tilde{a}\frac{v_x}{c^2}\right)-\{\tilde{a},0,0\}\right]+\frac{\tilde{j}d_x}{c^2}\vec{v} \tag{44}$$

Note that we need to define a new $oexp(t)$ function for this case. I will not provide the details here.

Getting the mass:

$$m' = m_0 \lim_{t\to 0}\frac{exp(t)}{oexp(t)} = m_0\gamma_v\left(1+\frac{\tilde{a}d_x}{c^2}\right)^{-1} = m\left(1+\frac{\tilde{a}d_x}{c^2}\right)^{-1} \tag{45}$$

A mass function for arbitrary time t:

$$mass(t) = m_0\frac{1}{\left(1+\frac{\tilde{a}\gamma_{\tilde{a}t}^3 pos_x(t)}{c^2}\right)\sqrt{1-\left(\frac{|\vec{vel}(t)|}{\left(1+\frac{\tilde{a}\gamma_{\tilde{a}t}^3 pos_x(t)}{c^2}\right)}\frac{1}{c}\right)^2}} \tag{46}$$

Defining momentum:

$$m\vec{o}m(t) = mass(t)\ \vec{vel}(t) \tag{47}$$

Momentum, again, is not affected by the observer's acceleration:

$$m\vec{o}m(0) \to m_0\gamma_{\vec{v}}\vec{v} = m\vec{v} \tag{48}$$

Differentiating momentum with respect to $exp(t)$ to get the force:

$$force_x(0) \;\to\; m_0\left[\left(1+\frac{\tilde{a}d_x}{c^2}\right)\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\gamma_{\vec{v}}^3 v_x + a_x\gamma_{\vec{v}}\right) - \tilde{a}\gamma_{\vec{v}}\right] \tag{49}$$

$$force_y(0) \;\to\; m_0\left[\left(1+\frac{\tilde{a}d_x}{c^2}\right)\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\gamma_{\vec{v}}^3 v_y + a_y\gamma_{\vec{v}}\right)\right] \tag{50}$$

$$force_z(0) \;\to\; m_0\left[\left(1+\frac{\tilde{a}d_x}{c^2}\right)\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\gamma_{\vec{v}}^3 v_z + a_z\gamma_{\vec{v}}\right)\right] \tag{51}$$

$$for\vec{c}e(0) \;\to\; m_0\left[\left(1+\frac{\tilde{a}d_x}{c^2}\right)\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\gamma_{\vec{v}}^3 \vec{v} + \gamma_{\vec{v}}\vec{a}\right) - \{\tilde{a},0,0\}\,\gamma_{\vec{v}}\right] \tag{52}$$

6. A final generalization

Now we will generalize the problem further. Until now we have dealt with a simple "boost": The case where the acceleration of the observer is only on one axis. Now we will solve the problem where the observer accelerates in some arbitrary direction. This acceleration would be dubbed $\vec{\tilde{a}}$.

Solving this case doesn't require any more fiddling with the Lorentz Transformation: We can just take the results from the "boost" case, and apply some linear algebra to generalize them. The results are listed below.

I have included another entry for the force, expressing it with respect to $\vec{F}$, the force acting on Obj in the frame of the non-accelerating observer. I have also included a legend, so this section may be used as reference.

$$\vec{vel}(0) \rightarrow \vec{v}\left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right) \tag{53}$$

$$\vec{acc}(0) \rightarrow \left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)\left[\vec{a}\left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)+\vec{v}\left(2\frac{\vec{\tilde{a}}\cdot\vec{v}}{c^2}\right)-\vec{\tilde{a}}\right]+\frac{\vec{\tilde{j}}\cdot\vec{d}}{c^2}\vec{v} \tag{54}$$

$$\vec{mom}(0) \rightarrow m_0\gamma_{\vec{v}}\vec{v} = m\vec{v} \tag{55}$$

$$mass(0) \rightarrow m_0\gamma_{\vec{v}}\left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)^{-1} = m\left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)^{-1} \tag{56}$$

$$\vec{force}(0) \rightarrow m_0\left[\left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\gamma_{\vec{v}}^3\vec{v}+\gamma_{\vec{v}}\vec{a}\right)-\vec{\tilde{a}}\gamma_{\vec{v}}\right] \tag{57}$$

$$\vec{force}(0) \rightarrow \left(1+\frac{\vec{\tilde{a}}\cdot\vec{d}}{c^2}\right)\vec{F}-m_0\gamma_{\vec{v}}\vec{\tilde{a}} \tag{58}$$

Legend:

$\vec{\tilde{a}}$ and $\vec{\tilde{j}}$: Respectively, the acceleration and jerk of the accelerating observer, as measured in the frame of the non-accelerating observer.

m, $\vec{d}$, $\vec{v}$, $\vec{a}$ and $\vec{F}$: Respectively, relativistic mass, position, velocity, acceleration and force of the object in the non-accelerating frame.

$\gamma_{\vec{v}}$: The gamma of $\vec{v}$, that is, $\left(1+\frac{|\vec{v}|^2}{c^2}\right)^{-0.5}$.

m_0: Rest mass of the object.

$\vec{vel}(0)$, $\vec{acc}(0)$, $\vec{mom}(0)$, etc.: Properties of the object as measured in the accelerating frame.

This reference table is the final product of this article.

A. Support for Assumption 1

All the equations that we arrived at were dependent of the assumption that we made. I reproduced it here:

Assumption 1: O and O' will agree about the coordinates of all events taking place at $time = 0$. That means that they will agree on the locations of all objects and the time reading on all clocks in the system.

What makes me think that this assumption is true?

It is hard to find discussion of this issue in the literature. In "Gravitation", the GR textbook by Thorne, Wheeler and Misner, there is a small section about analysis of an accelerating frame with Special Relativity:

> *"An accelerated observer can carry clocks and measuring rods with him, and can use them to set up a reference frame (coordinate system) in his neighborhood.*
> *His clocks, if carefully chosen so their structures are affected negligibly by acceleration (e.g., atomic clocks), will tick at the same rate as unaccelerated clocks moving momentarily along with him: [...] And his rods, if chosen to be sufficiently rigid, will measure the same length as momentarily comoving, unaccelerated rods do."*

This statement is equivalent to our Assumption 1.

Why is this assumption true? Why does velocity influence the "perception" of an observer, but acceleration doesn't? Why does an observer with velocity see rods shrinking but an observer with acceleration doesn't? Some reasoning can be made about this.

Imagine we are viewing an object moving about in the universe. We are observers analyzing it "from the outside" of it. If its velocity increases, its mass increases as well. But whatever acceleration it has, it has no effect on its mass. And if that object was a living observer, and we would ask him how the experience felt to him, "from the inside"? How would he say that the world changed when he picked up speed or acceleration?

I claim that there is a duality between his experiences from the inside and our measurements from the outside:[1]

	"From the outside"	"From the inside"
Velocity change	Mass of observer changed	*"I saw rods shrinking, clocks ticking slower."*
Acceleration change	Mass of observer didn't change	*"No rods shrunk and clocks ticked the same."*

This is, therefore, the explanation I propose: Because the acceleration of an object has no effect on its mass, the acceleration of an observer will not change his opinion of the locations of space-time events happening in the present time.

[1]The rods and clocks referred to in this table are those that, in the case of the velocity-changed observer, have the same location as the observer, and in the case of the acceleration-changed observer, have the same location and velocity as the observer.

REFERENCES

J. A. Wheeler, K. S. Thorne, C. W. Misner, Gravitation, W. H. Freeman, 1973

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